

# RELATIVE CARTIER DIVISORS AND K-THEORY

VIVEK SADHU AND CHARLES WEIBEL

**ABSTRACT.** We study the relative Picard group  $\mathrm{Pic}(f)$  of a map  $f : X \rightarrow S$  of schemes. If  $f$  is faithful affine, it is the relative Cartier divisor group  $\mathcal{I}(f)$ . The relative group  $K_0(f)$  has a  $\gamma$ -filtration, and  $\mathrm{Pic}(f)$  is the top quotient for the  $\gamma$ -filtration. When  $f$  is induced by a ring homomorphism  $A \rightarrow B$ , we show that the relative “nil” groups  $NPic(f)$  and  $NK_n(f)$  are continuous  $W(A)$ -modules.

## INTRODUCTION

If  $f : X \rightarrow S$  is a morphism of schemes, the relative Picard group  $\mathrm{Pic}(f)$  was defined by Bass in [1], and fits into a natural exact sequence

$$(0.1) \quad \mathcal{O}^\times(S) \xrightarrow{f^*} \mathcal{O}^\times(X) \xrightarrow{\partial} \mathrm{Pic}(f) \longrightarrow \mathrm{Pic}(S) \xrightarrow{f^*} \mathrm{Pic}(X).$$

The goal of this paper is to study this group as well as  $NPic(f)$ , defined to be  $\mathrm{Pic}(f[t])/\mathrm{Pic}(f)$ , where  $f[t] : X \times \mathbb{A}^1 \rightarrow S \times \mathbb{A}^1$ .

Our first observation is that when  $f$  is  $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$  for a commutative ring extension  $A \hookrightarrow B$ ,  $\mathrm{Pic}(f)$  is isomorphic to the relative Cartier divisor group  $\mathcal{I}(f)$ , defined in [13] as the group of invertible  $A$ -submodules of  $B$  under multiplication and studied in [15, 14, 16]. This definition of  $\mathcal{I}(f)$  also makes sense (and we still have  $\mathcal{I}(f) \cong \mathrm{Pic}(f)$ ) for scheme maps  $f : X \rightarrow S$  for which  $\mathcal{O}_S^\times \rightarrow f_*\mathcal{O}_X^\times$  is an injection of sheaves. It then follows from [16] that  $\mathrm{Pic}(f)$  is a contracted functor in the sense of Bass.

We then relate  $\mathrm{Pic}(f)$  to the relative group  $K_0(f)$ , which fits into an exact sequence

$$K_1(S) \xrightarrow{f^*} K_1(X) \xrightarrow{\partial} K_0(f) \longrightarrow K_0(S) \longrightarrow K_0(X).$$

For example, if  $f : A \hookrightarrow B$  is subintegral then  $K_0(f) \cong \mathrm{Pic}(f)$  (Proposition 2.5).

Let  $\mathcal{NI}$  denote the Zariski sheaf associated to the presheaf  $U \mapsto NPic(U, f^{-1}U)$  on  $S$ . In Theorem 4.1 and Theorem 4.7, we prove the following:

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**Theorem 0.2.** *Let  $f : X \rightarrow S$  be a faithful affine morphism of schemes.*

(1) *The Zariski sheaf  $\mathcal{N}\mathcal{I}$  is an étale sheaf on  $S$ . Moreover,*

$$NPic(f) \cong H_{\text{et}}^0(S, \mathcal{N}\mathcal{I}) = H_{\text{zar}}^0(S, \mathcal{N}\mathcal{I}).$$

(2) *If  $X$  and  $S$  are schemes then  $H_{\text{et}}^*(S, \mathcal{N}\mathcal{I}) \cong H_{\text{zar}}^*(S, \mathcal{N}\mathcal{I})$ .*

(3) *If  $X$  and  $S$  are both affine schemes then  $H_{\text{et}}^q(S, \mathcal{N}\mathcal{I}) = 0$  for  $q \neq 0$ .*

A secondary goal of this article is to study the relative  $K$ -theory groups  $K_n(f)$  associated to a morphism of schemes  $f : X \rightarrow S$ . By definition,  $K_n(f) = \pi_n K(f)$ , where  $K(f)$  is the homotopy fiber of  $K(S) \rightarrow K(X)$ . Comparing  $X \rightarrow S$  to  $X[t] \rightarrow S[t]$  yields groups  $NK_*(f)$ .

**Theorem 0.3.** *For each homomorphism  $f : A \rightarrow B$ :*

(1)  *$NK_n(f)$  is a continuous  $W(A)$ -module, for all  $n$ .*

(2)  *$NPic(f)$  is a continuous  $W(A)$ -module.*

(3)  *$\det : NK_0(f) \rightarrow NPic(f)$  is a  $W(A)$ -module homomorphism.*

(See Theorems 3.3 and 5.6, and Proposition 3.2). This implies that if  $\text{char}(A) = p > 0$  then both  $NK_n(f)$  and  $NPic(f)$  are  $p$ -groups, while if  $\text{char}(A) = 0$  the groups have the structure of  $A$ -modules.

We conclude with some remarks about  $K_n(f)$  when  $n$  is negative. If  $X$  and  $S$  have dimension at most  $d$ , then  $K_n(S) = K_n(X) = 0$  for  $n < -d$  in many cases. In such cases, it follows that  $K_n(f) = 0$  for  $n < -d - 1$ . The cohomological interpretation of the negative  $K$ -theory of a scheme in terms of the cdh-cohomology of the constant sheaf  $\mathbb{Z}$  is given in [4]. In the relative situation, we prove the following (Theorem 6.2 and Theorem 6.3):

**Theorem 0.4.** *Let  $f : X \rightarrow S$  be a finite morphism of  $d$ -dimensional noetherian schemes.*

(1) *If  $X$  and  $S$  are essentially of finite type over a field  $k$  of characteristic 0,  $K_{-d-1}(f) \cong H_{\text{cdh}}^d(S, f_*\mathbb{Z}/\mathbb{Z})$ .*

(2) *If  $\dim S = 1$ , then  $K_{-2}(f) \cong H_{\text{nis}}^1(S, f_*\mathbb{Z}/\mathbb{Z})$  and there is an extension*

$$0 \rightarrow H_{\text{nis}}^1(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) \rightarrow K_{-1}(f) \rightarrow H_{\text{nis}}^0(S, f_*\mathbb{Z}/\mathbb{Z}) \rightarrow 0.$$

## 1. RELATIVE Pic AND INVERTIBLE SUBMODULES

In [1], Bass defined  $\text{Pic}(f)$  to be the abelian group generated by  $[L_1, \alpha, L_2]$ , where the  $L_i$  are line bundles on  $S$  and  $\alpha : f^*L_1 \rightarrow f^*L_2$  is an isomorphism. The relations are:

- (1)  $[L_1, \alpha, L_2] + [L'_1, \alpha', L'_2] = [L_1 \otimes L'_1, \alpha \otimes \alpha', L_2 \otimes L'_2];$
- (2)  $[L_1, \alpha, L_2] + [L_2, \beta, L_3] = [L_1, \beta\alpha, L_3];$
- (3)  $[L_1, \alpha, L_2] = 0$  if  $\alpha = f^*(\alpha_0)$  for some  $\alpha_0 : L_1 \cong L_2$ .

[L, a]

*Remark 1.0.1.* By (1), every element of  $\text{Pic}(f)$  has the form  $[L, \alpha, \mathcal{O}_S]$ . Writing  $[L, \alpha]$  for  $[L, \alpha, \mathcal{O}_S]$ , an alternative presentation for  $\text{Pic}(f)$  is that it is generated by elements  $[L, \alpha]$  satisfying:  $[L, \alpha] + [L', \alpha'] = [L \otimes L', \alpha \otimes \alpha']$ ;  $[L, \alpha] = 0$  if (and only if) there is an isomorphism  $\alpha_0 : L \cong \mathcal{O}_S$  so that  $\alpha = f^*(\alpha_0)$ . It is easy to see, and observed by Bass, that the map  $\text{Pic}(f) \rightarrow \text{Pic}(S)$  sending  $[L, \alpha]$  to  $[L]$  fits into an exact sequence (0.1), where  $\partial(b) = [\mathcal{O}_S, b]$ .

hyper

**Proposition 1.1.** *Bass'  $\text{Pic}(f)$  is the hypercohomology group  $H^0(S, \mathcal{O}_S^\times \rightarrow f_*\mathcal{O}_X^\times)$ .*

*Proof.* Let  $C^*$  denote the mapping cone of  $\mathcal{O}_S^\times \rightarrow f_*\mathcal{O}_X^\times$ . A 0-cocyle of  $C^*$  is given by a cover  $\{U_i\}$  of  $S$ , a unit  $b_i$  of  $f^{-1}(U_i)$  for each  $i$ , and units  $a_{ij}$  of  $U_i \cap U_j$  for each  $i, j$  satisfying the cocyle condition (so that the  $\{a_{ij}\}$  define a line bundle  $L$  on  $S$ ) and such that  $b_i/b_j = f^\#(a_{ij})$  on each  $f^{-1}(U_i \cap U_j)$ . Since the  $\{b_i\}$  define an isomorphism  $f^*L \cong \mathcal{O}_X$ , each 0-cocyle defines an element  $\lambda = [L, \beta, \mathcal{O}_S]$  of  $\text{Pic}(f)$ . A 0-coboundary is given by  $a_{ij} = a_i/a_j$  and  $b_i = f^\#(a_i)$  for units  $a_i$  of  $U_i$ ; adding it to a cocyle does not change  $\lambda$ . Refining the cover does not change  $\lambda$  either. The result follows from the 5-lemma applied to the following diagram with exact rows (which is easily checked to be commutative):

$$\begin{array}{ccccccccc}
H^0(S, \mathcal{O}^\times) & \longrightarrow & H^0(X, \mathcal{O}^\times) & \longrightarrow & H^0(S, C^*) & \longrightarrow & H^1(S, \mathcal{O}^\times) & \longrightarrow & H^1(X, \mathcal{O}^\times) \\
\cong \downarrow & & \cong \downarrow & & \downarrow & & \cong \downarrow & & \cong \downarrow \\
\mathcal{O}^\times(S) & \longrightarrow & \mathcal{O}^\times(X) & \longrightarrow & \text{Pic}(f) & \longrightarrow & \text{Pic}(S) & \longrightarrow & \text{Pic}(X). \quad \square
\end{array}$$

Now suppose that  $f$  is faithful and affine. As observed in [16],  $\mathcal{I}(f)$  is isomorphic to  $H^0(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)$ . Thus Proposition 1.1 implies that  $\mathcal{I}(f) \cong \text{Pic}(f)$ . Here is a more elementary proof.

Pic=I

**Lemma 1.2.** *If  $f : X \rightarrow S$  is a faithful affine map, there is an isomorphism  $\rho : \mathcal{I}(f) \xrightarrow{\cong} \text{Pic}(f)$ , sending  $L$  to  $[L, i, \mathcal{O}_S]$ , where  $i : f^*L \cong \mathcal{O}_X$ .*

The isomorphism  $f^*L \cong \mathcal{O}_X$  is well defined, because in any affine open  $U = \text{Spec}(A)$  of  $S$  we have  $f^{-1}U = \text{Spec}(B)$  with  $A \subset B$ ; it was proven by Roberts and Singh [13] that  $L \subset B$  induces  $L \otimes_A B \cong B$ .

*Proof.* Since  $\rho(LL') = [L \otimes L', i \otimes i', \mathcal{O}_S] = [L, i, \mathcal{O}_S] + [L', i', \mathcal{O}_S]$ ,  $\rho$  is a homomorphism. To define the inverse map, we use the presentation of  $\text{Pic}(f)$  and the observation that because  $\mathcal{O}_S \rightarrow f_*\mathcal{O}_X$  is an injection, so is  $L \rightarrow L \otimes f_*\mathcal{O}_X$  for every line bundle  $L$ . Given a triple  $[L_1, \alpha, L_2]$ , we set  $L = L_2^{-1} \otimes L_1$ , so that  $\alpha$  induces an isomorphism  $f^*L \cong f^*(L_2)^{-1} \otimes f^*(L_1) \cong \mathcal{O}_X$ , and define  $\psi([L_1, \alpha, L_2])$  to be the submodule  $L$  of  $L \otimes f_*\mathcal{O}_X \cong f_*\mathcal{O}_X$ . Since  $\psi$  is compatible with the relations of  $\text{Pic}(f)$ , it descends to a homomorphism  $\psi : \text{Pic}(f) \rightarrow \mathcal{I}(f)$ . Since  $[L_1, \alpha, L_2] = [L_2^{-1} \otimes L_1, \alpha, \mathcal{O}_S]$  in  $\text{Pic}(f)$  and  $f^*(L) = \mathcal{O}_X$  for all  $L \in \mathcal{I}(f)$ ,  $\psi$  is an inverse to  $\rho$ .  $\square$

## 2. RELATIVE $K_0$ AND $\text{Pic}$

Bass gave a presentation of a relative group  $K_0(f)$  associated to  $f : A \rightarrow B$  in [1] and [2, VII.5]; see [29, II.2.10]. It is generated by triples  $[P_1, \alpha, P_2]$ , where the  $P_i$  are finitely generated projective  $A$ -modules (or vector bundles on  $S$ ) and  $\alpha$  is an isomorphism  $f^*(P_1) \xrightarrow{\cong} f^*(P_2)$ , and agrees with the group  $\pi_0 K(f)$  of [29, IV.1.11]. The relations are:

- (1)  $[P_1, \alpha, P_2] + [P'_1, \alpha', P'_2] = [P_1 \oplus P'_1, \alpha \oplus \alpha', P_2 \oplus P'_2]$ ,
- (2)  $[P_1, \alpha, P_2] + [P_2, \beta, P_3] = [P_1, \beta\alpha, P_3]$ ,
- (3)  $[P_1, \alpha, P_2] = 0$  if  $\alpha = f^*(\alpha_0)$  for some  $\alpha_0 : P_1 \cong P_2$ .

By (1), every element of  $K_0(f)$  has the form  $[P, \alpha, A^n]$ .

Bass showed [2, VII.5.3] that there is an exact sequence for each  $f : A \rightarrow B$ :

$$\boxed{\text{seq:K0}} \quad (2.1) \quad K_1(A) \xrightarrow{f^*} K_1(B) \xrightarrow{\partial} K_0(f) \longrightarrow K_0(A) \longrightarrow K_0(B),$$

where for  $g \in GL_n(B)$  we have  $\partial([g]) = [A^n, g, A^n]$ . Since we do not know if the corresponding sequence is exact for a quasi-projective map  $f : X \rightarrow S$ , we will restrict to the affine case in this section and the next.

**excision** **Lemma 2.2** (Excision). *Let  $f : A \rightarrow B$  be a ring homomorphism, and  $I$  is an ideal of  $A$  mapping isomorphically onto an ideal of  $B$ ; write  $\bar{f} : A/I \subset B/I$  for the induced map. Then excision holds for  $K_n$  for all  $n \leq 0$ :  $K_n(f) \cong K_n(\bar{f})$ .*

*Proof.* It suffices to consider the case  $n = 0$ . Because  $K_0(A, I) \cong K_0(B, I)$  [29, Ex. II.2.3] and  $K_1(A, I) \rightarrow K_1(B, I)$  is onto [29, III.2.2.1], the double-relative group

vanishes:  $K_0(A, B, I) = 0$ . Applying contraction, we also have  $K_{-1}(A, B, I) = 0$ . The result now follows from the exact sequence

$$K_0(A, B, I) \rightarrow K_0(f) \rightarrow K_0(\bar{f}) \rightarrow K_{-1}(A, B, I). \quad \square$$

*Remark.* The failure of Lemma 2.2 in the non-affine setting was investigated in [12, A.5–6]. For example, if  $X$  is the normalization of  $S$  and the support  $Y$  of the conductor  $\mathfrak{c}$  is 1-dimensional, the obstruction is  $K_0(S, X, Y) \cong H^1(Y, \mathfrak{c}/\mathfrak{c}^2 \otimes \Omega_{X/S})$ .

As observed by Bass and Murthy long ago [3], the determinant  $K_0(S) \rightarrow \text{Pic}(S)$  induces a surjective homomorphism

eq:det

$$(2.3) \quad \det : K_0(f) \rightarrow \text{Pic}(f), \quad \det[P_1, \alpha, P_2] = [\det(P_1), \det(\alpha), \det(P_2)].$$

Since  $SK_0(S)$  is the kernel of  $\det : K_0(S) \rightarrow \text{Pic}(S)$ , we write  $SK_0(f)$  for the kernel of  $\det : K_0(f) \rightarrow \text{Pic}(f)$ .

Recall [29, II.4.2] that a  $\lambda$ -ring  $K = \mathbb{Z} \oplus \tilde{K}$  has a *positive structure* if it contains a  $\lambda$ -semiring  $P$  (positive elements) including  $\mathbb{N}$ , such that every element of  $\tilde{K}$  can be written as a difference of positive elements, the augmentation  $\epsilon : K \rightarrow \mathbb{Z}$  sends  $P$  to  $\mathbb{N}$  and, if  $p \in P$  has  $\epsilon(p) = n$ , then  $\lambda^i p = 0$  for  $i > n$  and  $\lambda^n p$  is a unit. The *line elements* are  $\{p \in P : \epsilon(p) = 1\}$ ; they form a subgroup of the units of  $K$ .

lambda-ops

**Proposition 2.4.** *Let  $f : A \rightarrow B$  be a homomorphism of commutative rings. The operations  $\lambda^i[P_1, \alpha, P_2] = [\Lambda^i P_1, \Lambda^i \alpha, \Lambda^i P_2]$  give  $\mathbb{Z} \oplus K_0(f)$  the structure of a  $\lambda$ -ring with a positive structure. The top two ideals in the  $\gamma$ -filtration are  $F_\gamma^1 = \tilde{K}_0$  and  $F_\gamma^2 = SK_0(f)$ , and the group of its line elements is  $\text{Pic}(f) \cong F_\gamma^1/F_\gamma^2$ .*

*Proof.* Given  $f : A \rightarrow B$ , choose a surjection  $\pi : \mathbb{Z}[X] \rightarrow B$  from a polynomial ring  $\mathbb{Z}[X]$  in many variables to  $B$ ; let  $R$  be the pullback ring  $R = \{(p, a) \in \mathbb{Z}[X] \times A : \pi(p) = f(a)\}$ , with  $\tilde{f} : R \rightarrow \mathbb{Z}[X]$  the projection. Since  $K_1(\mathbb{Z}[X]) = \pm 1$  and  $K_0(\mathbb{Z}[X]) = \mathbb{Z}$ , we have  $K_0(\tilde{f}) \xrightarrow{\cong} \tilde{K}_0(R)$ , and this map is compatible with the operations  $\lambda^i$ . Similarly, we have  $\text{Pic}(\tilde{f}) \cong \text{Pic}(R)$ . By Excision 2.2 for  $K_0$  and  $\text{Pic}$ ,  $K_0(\tilde{f}) \cong K_0(f)$  and  $\text{Pic}(\tilde{f}) \cong \text{Pic}(f)$ . Hence  $\mathbb{Z} \oplus K_0(f) \cong \mathbb{Z} \oplus \tilde{K}_0(R)$  is a  $\lambda$ -ring. Thus the result follows from the fact that the operations  $\lambda^i$  make  $K_0(R)$  into a  $\lambda$ -ring, with  $F_\gamma^2 = SK_0(R)$ , and  $\tilde{K}_0(R)/SK_0(R) \cong \text{Pic}(R)$ .  $\square$

Recall (Swan [17]) that an extension  $A \subset B$  is said to be *subintegral* if  $B$  is integral over  $A$ , and  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is a bijection inducing isomorphisms on all residue fields.

**subint**

**Proposition 2.5.** (*Ischebeck*) *If  $f : A \hookrightarrow B$  is subintegral then  $K_0(f) \cong \text{Pic}(f)$ ,  $K_n(f) = 0$  for all  $n < 0$ , and there is an exact sequence*

$$1 \rightarrow B^\times/A^\times \rightarrow K_0(f) \rightarrow K_0(A) \rightarrow K_0(B) \rightarrow 0.$$

*Proof.* When  $A \subset B$  is subintegral, Ischebeck proved in [9, Prop. 7] that the natural map  $K_0(A) \rightarrow K_0(B)$  is surjective and  $SK_1(A) \rightarrow SK_1(B)$  is onto, so the cokernel of  $K_1(A) \rightarrow K_1(B)$  is  $B^\times/A^\times$ . The exact sequence follows from (2.1). Finally, Ischebeck proved in [9, p. 331] that the determinant (2.3) induces an isomorphism from the kernel of  $K_0(A) \rightarrow K_0(B)$  onto the kernel of  $\text{Pic}(A) \rightarrow \text{Pic}(B)$ . The result now follows from (2.1).

Replacing  $A$  and  $B$  by Laurent polynomial extensions, the Fundamental Theorem of  $K$ -theory [29, III.4.1] implies that  $LK_n(f) \cong K_{n-1}(f)$  and  $K_{-1}(f) \cong L\text{Pic}(f)$ . Since  $A[t, 1/t] \subset B[t, 1/t]$  is subintegral, we have  $L\text{Pic}(f) = 0$  by Proposition 5.6 of [16]. This shows that  $K_n(f) = 0$  for all  $n < 0$ .  $\square$

Given an extension  $f : A \hookrightarrow B$ , let  $i : A \hookrightarrow {}^+A$  be the seminormalization of  $A$  in  $B$  and  ${}^+f : {}^+A \hookrightarrow B$  the induced map. There is an exact sequence

$$\cdots \rightarrow K_n(i) \rightarrow K_n(f) \rightarrow K_n({}^+f) \rightarrow K_{n-1}(i) \rightarrow \cdots$$

**Corollary 2.6.**  $K_n(f) \xrightarrow{\cong} K_n({}^+f)$  for  $n < 0$ , and the following sequence is exact.

$$0 \rightarrow K_0(i) \rightarrow K_0(f) \rightarrow K_0({}^+f) \rightarrow 0.$$

*Proof.* By Proposition 2.5 and [16, Lemma 3.3], the map  $K_0(i) \cong \text{Pic}(i) \rightarrow \text{Pic}(f)$  is an injection. Since it factors through  $K_0(i) \rightarrow K_0(f)$ , the latter map is an injection. Since  $K_n(i) = 0$  for  $n < 0$ , again by Proposition 2.5, we are done.  $\square$

### 3. THE $W(A)$ -MODULE STRUCTURE ON $NK_0(f)$ AND $NPic(f)$

In this section, we fix a ring homomorphism  $f : A \rightarrow B$  and show that  $NK_0(f)$  and  $NPic(f)$  are continuous modules over the ring  $W(A)$  of big Witt vectors, so that

**seq:NK0**

$$(3.1) \quad NK_1(A) \rightarrow NK_1(B) \xrightarrow{\partial} NK_0(f) \rightarrow NK_0(A) \rightarrow NK_0(B)$$

is a sequence of  $W(A)$ -modules. Recall that  $(1 + tA[[t]])^\times$  is the underlying abelian group of the ring  $W(A)$ ; a  $W(A)$ -module is continuous if every element is killed by one of these ideals  $(1 + t^n A[[t]])^\times$ .

We first recall the continuous  $W(R)$ -module structure on  $NK_*(A)$  when  $R$  is commutative and  $A$  is an  $R$ -algebra, due to Stienstra [18]. As  $NK_*(A)$  is a continuous module, it suffices to describe multiplication by  $(1 - rt^m)$ ,  $r \in R$ . Setting  $S = R[s]/(s^m - r)$ , the inclusion  $i : R \subset S$  induces a base change functor  $i^* : \mathbf{P}(A[t]) \rightarrow \mathbf{P}(A \otimes_R S[t])$  and a transfer map  $i_* : \mathbf{P}(A \otimes_R S[t]) \rightarrow \mathbf{P}(A[t])$ . If  $\sigma$  denotes the  $S$ -algebra map  $S[t] \rightarrow S[t]$ ,  $\sigma(t) = st$ , then the composition  $F = i_* \sigma^* i^*$  is an additive self-functor of  $\mathbf{P}(A[t])$ . As noted in [26, 1.5], the composition  $\mathbf{P}(A) \rightarrow \mathbf{P}(A[t]) \xrightarrow{F} \mathbf{P}(A[t]) \rightarrow \mathbf{P}(A)$  is  $\otimes_R S$ , so  $F$  induces multiplication by  $m$  on the summand  $K_*(A)$  of  $K_*(A[t])$ ; the restriction of  $F$  to  $NK_*(A)$  is multiplication by  $(1 - rt^m)_*$ . If  $A \rightarrow B$  is an  $R$ -algebra map,  $NK_*(A) \rightarrow NK_*(B)$  is a homomorphism of continuous  $W(R)$ -modules.

We can adapt these formulas to define a multiplication by  $(1 - at^m)_*$  on  $K_0(f)$  and  $NK_0(f)$  when  $a \in A$ : send  $[P_1, \alpha, P_2]$  to  $[F(P_1), F(\alpha), F(P_2)]$ . It is clear from (2.1) that  $(1 - at^m)_*$  is compatible with the exact sequence (3.1). A priori, though, the maps  $(1 - at^m)_*$  do not fit together to make  $NK_0(f)$  into a  $W(A)$ -module.

**ctn-W** **Proposition 3.2.** *For any homomorphism  $f : A \rightarrow B$ ,  $NK_0(f)$  is a continuous  $W(A)$ -module, and (3.1) is an exact sequence of continuous  $W(A)$ -modules.*

*Proof.* As in the proof of Proposition 2.4, write  $B = \mathbb{Z}[X]/I$ , where  $\mathbb{Z}[X]$  is a polynomial ring. Let  $R$  denote the pullback ring  $A \times_B \mathbb{Z}[X]$ , and write  $\tilde{f} : R \rightarrow \mathbb{Z}[X]$  for the quotient map. Since  $NK_*(\mathbb{Z}[X]) = 0$ , we have  $NK_n(\tilde{f}) \cong NK_n(R)$  for all  $n$ . Since  $A = R/I$ , Lemma 2.2 and [25] imply that the groups  $NK_0(f) \cong NK_0(\tilde{f}) \cong NK_0(R)$  are continuous  $W(R)$ -modules.

Since  $W(A) = W(R)/W(I)$ , where  $W(I) = 1 + tI[[t]]$ , we are reduced to showing that  $(1 - rt^m)$  acts as zero on  $K_0(f)$  whenever  $r \in I$ . When  $r$  is in the kernel  $I$  of  $R \rightarrow A$ , the ring  $A \otimes_R S$  is just  $A[s]/(s^m)$ , so  $(1 - rt^m)$  and  $(1 - 0t^m)$  act identically on  $K_0(f[t])$ . This shows that  $(1 - rt^m)$  acts as zero on  $K_0(f)$  and proves that the action of  $W(A)$  on  $K_0(f)$  is well defined and continuous.  $\square$

Applying  $N$  to the determinant described in (2.3), we get an exact sequence

$$0 \rightarrow NSK_0(f) \rightarrow NK_0(f) \xrightarrow{\det} \mathrm{NPic}(f) \rightarrow 0.$$

If  $[P, \alpha, A[t]^n]$  is in  $NK_0(f)$  then  $\det[P, \alpha, A[t]^n] = [\det(P), \det(\alpha), A[t]]$ .

**NI** **Theorem 3.3.** *For any homomorphism  $f : A \rightarrow B$ ,  $\mathrm{NPic}(f)$  is a continuous  $W(A)$ -module, and  $\det : NK_0(f) \rightarrow \mathrm{NPic}(f)$  is a  $W(A)$ -module homomorphism.*

*Proof.* Since the group  $NK_0(f)$  is a continuous  $W(A)$ -module by Proposition 3.2, it is enough to show that  $NSK_0(f)$  is closed under multiplication by  $W(A)$ . Since every element of  $W(A)$  can be written as  $\prod_{m>0}(1 - a_m t^m)$ , with  $a_m \in A$ , and for any element  $u$  of  $NK_0(f)$  there is an  $n$  so that  $\prod_{m \geq n}(1 - a_m t^m) * u = 0$ , it is enough to show that  $NSK_0(f)$  is closed under multiplication by  $(1 - at^m)$  for any  $a \in A$  and  $m \geq 1$ .

It is enough to show that  $F = i_* \sigma^* i^*$  sends  $SK_0(f[t])$  to itself. We now modify the argument of [5, 4.1]. Fix  $u = [P, \alpha, A[t]^n]$  in  $SK_0(f[t])$ ; By Remark 1.0.1,  $\det(u) = 0$  implies that  $\det(P) = A[t]$  and  $\det(\alpha) \in A$ . By naturality of  $\det$ ,  $\sigma^* i^*(u) = [P \otimes S, \alpha \otimes S, S[t]^n]$ ,  $\det(P \otimes S) = S[t]$ ,  $\det(\alpha \otimes S) \in S$  and  $F(u) = [i_*(P \otimes S), i_*(\alpha \otimes S), A[t]^n]$ . By Corollary 3.2 of [5] applied to  $A[t] \subset S[t]$ ,  $\det(i_*(P \otimes S)) = A[t]$  and  $\det(\alpha \otimes S) = \det(\alpha)^m \in A$ , so  $\det(F(u)) = 0$ .  $\square$

**Corollary 3.4.** *If  $\text{char}(A) = p$  then  $NPic(f)$  is a  $p$ -group.*

*If  $\mathbb{Q} \subseteq A$  then  $NPic(f)$  is an  $A$ -module.*

*Proof.* Any continuous  $W(A)$ -module has these properties; see [25, 3.3].  $\square$

#### 4. SHEAF PROPERTIES OF $NPic(f)$

When  $f : X \rightarrow S$  is a faithful affine morphism of schemes, let  $\mathcal{I}(f)_{\text{zar}}$  denote the Zariski sheaf  $f_* \mathcal{O}_X^\times / \mathcal{O}_S^\times$  on the category  $Sm/S$  of smooth schemes over  $S$ ; by [16, 4.4],  $\mathcal{I}(f)_{\text{zar}}$  is also an étale sheaf, and  $H_{\text{et}}^0(S, \mathcal{I}(f)_{\text{zar}}) = H_{\text{nis}}^0(S, \mathcal{I}(f)_{\text{zar}}) = \text{Pic}(f)$ . Our choice of  $Sm/S$  is dictated by the need to not only include étale extensions but be closed under product with  $\mathbb{A}_S^1 \xrightarrow{\pi} S$ .

Let  $\pi^* \mathcal{I}(f)$  denote the restriction of  $\mathcal{I}(f)_{\text{zar}}$  to  $Sm/\mathbb{A}_S^1$  along  $\pi$ . Its direct image  $\pi_*(\pi^* \mathcal{I}(f))$  is the Zariski sheaf  $\mathcal{I}(f)_{\text{zar}} \oplus \mathcal{N}\mathcal{I}(f)$  on  $Sm/S$ , where  $\mathcal{N}\mathcal{I}(f)$  denotes the Zariski sheaf on  $Sm/S$  associated to the presheaf  $U \mapsto NPic(f \times_S U)$ .

**Theorem 4.1.** *Let  $f : X \rightarrow S$  be a faithful affine morphism of schemes. Then  $\mathcal{N}\mathcal{I}(f)$  is an étale sheaf on  $S$ . Moreover,*

$$H_{\text{et}}^0(S, \mathcal{N}\mathcal{I}(f)) = H_{\text{zar}}^0(S, \mathcal{N}\mathcal{I}(f)) = NPic(f).$$

*Proof.* Since  $\pi^* \mathcal{I}(f)$  is an étale sheaf on  $\mathbb{A}_S^1$ , its direct image  $\pi_* \pi^* \mathcal{I}(f)$  is an étale sheaf on  $S$ ; since  $\pi_* \pi^* \mathcal{I}(f) \cong \mathcal{I}(f)_{\text{zar}} \oplus \mathcal{N}\mathcal{I}(f)$ ,  $\mathcal{N}\mathcal{I}(f)$  is also an étale sheaf. Since

$$H_{\text{et}}^0(S, \pi_* \pi^* \mathcal{I}(f)) = H_{\text{et}}^0(\mathbb{A}_S^1, \pi^* \mathcal{I}(f)) = \text{Pic}(f[t]) = \text{Pic}(f) \oplus NPic(f),$$



we see that  $H_{\text{et}}^0(S, \mathcal{N}\mathcal{I}(f)) = \text{NPic}(f)$ . If  $S_s$  is a Zariski local scheme of  $S$ , this shows that the stalk  $\mathcal{N}\mathcal{I}(f)_s = H_{\text{zar}}^0(S_s, \mathcal{N}\mathcal{I}(f))$  equals  $H_{\text{et}}^0(S_s, \mathcal{N}\mathcal{I}(f))$ .  $\square$

**snormal**

**Example 4.2.** If  $f$  is seminormal, the sheaf  $\mathcal{N}\mathcal{I}(f)$  vanishes and  $\text{NPic}(f) = 0$ . This follows from Theorem 4.1 and [15, 1.5], which states that  $\text{NPic}(A, B) = 0$  when  $A$  is seminormal in  $B$ .

We now modify an argument of Vorst [22] and van der Kallen [21]. Suppose that  $\text{Spec}(A) = \bigcup_{i=0}^r U_i$ , where  $U_i = \text{Spec}(A_{s_i})$ . Given a presheaf  $F$  of abelian groups on  $\text{Spec}(A)$ , we write  $C^\bullet(\{U_i\}, F)$  for the augmented Čech complex:

$$0 \rightarrow F(A) \xrightarrow{\epsilon} \prod_{i=0}^r F(A_{s_i}) \rightarrow \prod_{0 \leq i < j \leq r} F(A_{s_i s_j}) \rightarrow \cdots \rightarrow F(A_{s_0 s_1 \cdots s_r}) \rightarrow 0.$$

Given  $s \in A$ , we have an  $A$ -algebra map  $\sigma : A[x] \rightarrow A[x]$  determined by  $\sigma(x) = sx$ . We write  $NF(A)_{[s]}$  for the direct limit of  $F(A[x]) \xrightarrow{\sigma} F(A[x]) \xrightarrow{\sigma} \cdots$ . Suppose that for all  $0 \leq i_0 < i_1 < \cdots < i_p \leq r$  and  $j \leq p$ :

**NPIC**

$$(4.3) \quad NF(A_{s_{i_0} \cdots s_{i_j} \cdots s_{i_p}}[x]) \cong NF(A_{s_{i_0} \cdots \hat{s}_{i_j} \cdots s_{i_p}}[x])_{[s_{i_j}]}.$$

In this situation, Vorst proved [22, 1.2] that the sequence  $C^\bullet(\{U_i\}, NF)$  is always exact. He also proved that  $F = NK_n$  satisfied (4.3), so that  $C^\bullet(\{U_i\}, NK_n)$  is exact for all  $n$ . (See [22, 1.4] or [29, V.8.5]; the nonzerodivisor hypothesis is unnecessary by [20].)

**cech-NU**

*Remark 4.4.* It is easy to see (and follows from Vorst's result [22, 1.2]) that the functor  $NU(A) = (A[t])^\times / A^\times$  satisfies (4.3). From the exact sequence of complexes

$$0 \rightarrow C^\bullet(\{U_i\}, NU) \rightarrow C^\bullet(\{U_i\}, NU(- \otimes_A B)) \rightarrow C^\bullet(\{U_i\}, NU(- \otimes_A B)/NU) \rightarrow 0$$

we see that  $C^\bullet(\{U_i\}, F)$  is also exact for the functor  $F(A_s) = NU(B_s)/NU(A_s)$ .

**cech-A**

**Lemma 4.5.**  $C^\bullet(\{U_i\}, \text{NPic})$  is always an exact sequence.

*Proof.* By Theorem 4.2 of [27], given  $s \in A$  we have  $\text{NPic}(A_s) \cong \text{NPic}(A)_{[s]}$  and hence  $\text{NPic}(A_s[x]) \cong \text{NPic}(A[x])_{[s]}$ . This implies that  $\text{NPic}$  satisfies (4.3). Vorst's result shows that  $C^\bullet(\{U_i\}, \text{NPic})$  is an exact sequence.  $\square$

We apply these considerations to the presheaf  $\text{NPic}(f) : U \mapsto \text{NPic}(f|_U)$ .

**cech**

**Lemma 4.6.** *Suppose that  $\mathrm{Spec}(A) = \cup_{i=0}^n U_i$ , where  $U_i = \mathrm{Spec}(A_{s_i})$ . If  $f : A \hookrightarrow B$  is a ring extension, the complex  $C^\bullet(\{U_i\}, \mathrm{NPic}(f))$  is exact.*

$$0 \rightarrow \mathrm{NPic}(A, B) \rightarrow \prod_{i=0}^n \mathrm{NPic}(A_{s_i}, B_{s_i}) \rightarrow \prod_{i_1 < i_2} \mathrm{NPic}(A_{s_{i_1 s_{i_2}}}, B_{s_{i_1 s_{i_2}}}) \rightarrow \cdots$$

*Proof.* Let  ${}^+A$  denote the subintegral closure of  $A$  in  $B$ , so  ${}^+A$  is seminormal in  $B$  and we have  $A \subset {}^+A \subset B$ . By [14, Prop. 4.1], we have an exact sequence

$$1 \rightarrow \mathrm{NPic}(A, {}^+A) \rightarrow \mathrm{NPic}(A, B) \rightarrow \mathrm{NPic}({}^+A, B) \rightarrow 1.$$

By Example 4.2, the third term vanishes and we have  $\mathrm{NPic}(A, {}^+A) \cong \mathrm{NPic}(A, B)$ . Thus we may assume that  $B$  is subintegral over  $A$ . In this case, Ischebeck proved [9, Prop. 7] that  $\mathrm{NPic}(A) \rightarrow \mathrm{NPic}(B)$  is surjective. Now the result follows from Remark 4.4, Lemma 4.5 and the long exact cohomology sequences associated to

$$\begin{aligned} 0 \rightarrow C^\bullet(\{U_i\}, F) \rightarrow C^\bullet(\{U_i\}, \mathrm{NPic}(f)) \rightarrow C^\bullet(\{U_i\}, \mathrm{NPic}(f)/F) \rightarrow 0, \\ 0 \rightarrow C^\bullet(\{U_i\}, \mathrm{NPic}(f)/F) \rightarrow C^\bullet(\{U_i\}, \mathrm{NPic}) \rightarrow C^\bullet(\{f^{-1}(U_i)\}, \mathrm{NPic}) \rightarrow 0. \quad \square \end{aligned}$$

**quasi**

**Theorem 4.7.** *Let  $f : A \hookrightarrow B$  be an extension of rings. Then:*

$$H_{\mathrm{et}}^q(\mathrm{Spec}(A), \mathcal{N}\mathcal{I}) = \begin{cases} \mathrm{NPic}(f) & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

*Proof.* The case  $q = 0$  is given by Theorem 4.1. By Lemma 4.6, the Čech cohomology groups  $\check{H}^q(\mathrm{Spec}(A), \mathcal{N}\mathcal{I})$  vanish for  $q > 0$ . Using the Cartan criterion [11, III.2.17],  $H_{\mathrm{et}}^q(\mathrm{Spec}(A), \mathcal{N}\mathcal{I})$  equals  $\check{H}^q(\mathrm{Spec}(A), \mathcal{N}\mathcal{I}) = 0$  for  $q > 0$ .  $\square$

**et=zar**

**Corollary 4.8.** *Let  $f : X \rightarrow S$  be a faithful affine morphism of schemes. Then*

$$H_{\mathrm{et}}^*(S, \mathcal{N}\mathcal{I}) \cong H_{\mathrm{zar}}^*(S, \mathcal{N}\mathcal{I}).$$

*Proof.* Consider the site change map  $\tau : S_{\mathrm{et}} \rightarrow S_{\mathrm{zar}}$ . Then by Theorem 4.7, the higher direct image sheaves  $R^q \tau_* \mathcal{N}\mathcal{I}$  vanish for  $q > 0$ . Therefore the Leray spectral sequence degenerates, yielding the result.  $\square$

*Remark.* More generally, if  $f : X \rightarrow S$  is any morphism of schemes then  $\mathcal{O}_S^\times$  may not inject into  $f_* \mathcal{O}_X^\times$ . In this case, if we interpret  $f_* \mathcal{O}_X^\times / \mathcal{O}_S^\times$  as the mapping cone of  $\mathcal{O}_S^\times \rightarrow f_* \mathcal{O}_X^\times$  (a complex of Zariski sheaves) and use sheaf hypercohomology, then Theorem 4.1 remains valid. However, Theorem 4.7 may fail in this setting.

5. MODULE STRUCTURES ON  $NK_n(f)$ **sec:module**

Given an exact functor  $F : \mathcal{P} \rightarrow \mathcal{Q}$ , the relative  $K$ -theory groups  $K_n(F)$  fit into an exact sequence

$$\cdots \xrightarrow{F} K_{n+1}\mathcal{Q} \xrightarrow{\partial} K_n(F) \rightarrow K_n\mathcal{P} \xrightarrow{F} K_n\mathcal{Q} \xrightarrow{\partial} \cdots$$

ending in  $K_0\mathcal{Q} \xrightarrow{\partial} K_{-1}(F)$ . Waldhausen showed that the  $K_n(F)$  are the homotopy groups  $\pi_{n+2}|wS_\bullet(S_\bullet F)|$  ( $n \geq 0$ ), where  $S_n F$  denotes the category of pairs

$$(P_*, Q_*) = (P_1 \rightrightarrows P_2 \rightrightarrows \cdots \rightrightarrows P_n, Q_0 \rightrightarrows Q_1 \rightrightarrows \cdots \rightrightarrows Q_n)$$

( $P_i \in \mathcal{P}$  and  $Q_j \in \mathcal{Q}$ ), together with choices of  $Q_i/Q_j$  for  $i > j$ , such that  $F(P_*)$  is  $Q_1/Q_0 \rightrightarrows \cdots \rightrightarrows Q_n/Q_0$ . (See [23, 1.5.4–7] or [29, IV.8.5.3].)

**ex:f[t,1/t]**

**Example 5.1.** If  $A$  is a ring, we write  $\mathbf{P}(A)$  for the category of finitely generated projective  $A$ -modules. Given a ring homomorphism  $f : A \rightarrow B$ , we have an exact functor  $\mathbf{P}(f) : \mathbf{P}(A) \rightarrow \mathbf{P}(B)$ ; by abuse, we write  $K_*(f)$  for  $K_*\mathbf{P}(f)$ . Writing  $f[t]$  for  $A[t] \rightarrow B[t]$ , we have  $K_*(f[t]) = K_*(f) \oplus NK_*(f)$ . The Fundamental Theorem of  $K$ -theory easily extends to the relative setting, yielding

$$K_*(f[t, 1/t]) \cong K_*(f) \oplus NK_*(f) \oplus NK_*(f) \oplus K_{*-1}(f).$$

Let  $A$  be a commutative ring. As in [29], we write  $\mathbf{End}(A)$  for the category of pairs  $(P, \alpha)$ , where  $P$  in  $\mathbf{P}(A)$  and  $P \xrightarrow{\alpha} P$  is an endomorphism, and write  $\mathbf{Nil}(A)$  for the full subcategory of  $\mathbf{End}(A)$  consisting of all  $(P, \alpha)$  with  $\alpha$  nilpotent. As pointed out in [29, II.7.4],  $K_*\mathbf{End}(A) \cong K_*(A) \oplus \text{End}_*(A)$  and  $K_*\mathbf{Nil}(A) \cong K_*(A) \oplus \text{Nil}_*(A)$ , where  $\text{End}_*(A)$  is a graded-commutative ring and  $\text{Nil}_*(A)$  is a graded  $\text{End}_*(A)$ -module. By naturality, the exact functors  $\mathbf{Nil}(f) : \mathbf{Nil}(A) \rightarrow \mathbf{Nil}(B)$  yield relative groups  $K_*\mathbf{Nil}(f) \cong K_*(f) \oplus \text{Nil}_*(f)$ .

The category  $\mathbf{Nil}(A)$  is equivalent to the category  $\mathbf{H}_{1,t}(A[t])$  of  $t$ -primary torsion  $A[t]$ -modules  $M$  with  $pd_{A[t]} M = 1$ . Specifically, if  $(P, \nu)$  is in  $\mathbf{Nil}(A)$ , and we write  $P_\nu$  for the  $A[t]$ -module  $P$  on which  $t$  acts as  $\nu$ , then  $P_\nu$  has projective dimension 1 over  $A[t]$ . The Fundamental Theorem ([29, V.8.2]) implies that  $\text{Nil}_n(A) \cong NK_{n+1}(A)$ . We also have  $K\mathbf{P}(A[t]) \cong K\mathbf{H}(A[t])$  (see e.g., [29, V.3.2]).

**Nil=NK**

**Proposition 5.2.** *There is a natural isomorphism  $\text{Nil}_n(f) \cong NK_{n+1}(f)$ .*

*Proof.* From the diagram of exact categories

$$\begin{array}{ccccccccc} \mathbf{Nil}(A) & \xrightarrow{\cong} & \mathbf{H}_{1,t}(A[t]) & \longrightarrow & \mathbf{H}(A[t]) & \xleftarrow{\cong} & \mathbf{P}(A[t]) & \longrightarrow & \mathbf{P}(A[t, 1/t]) \\ \downarrow & & & & & & \downarrow & & \downarrow \\ \mathbf{Nil}(B) & \xrightarrow{\cong} & \mathbf{H}_{1,t}(B[t]) & \longrightarrow & \mathbf{H}(B[t]) & \xleftarrow{\cong} & \mathbf{P}(B[t]) & \longrightarrow & \mathbf{P}(B[t, 1/t]) \end{array}$$

we get a fibration sequence of  $K$ -theory spectra

$$\begin{array}{ccccc} K\mathbf{Nil}(A) & \longrightarrow & K(A[t]) & \longrightarrow & K(A[t, 1/t]) \\ \downarrow & & \downarrow f[t]^* & & \downarrow f[t, 1/t]^* \\ K\mathbf{Nil}(B) & \longrightarrow & K(B[t]) & \longrightarrow & K(B[t, 1/t]). \end{array}$$

Taking vertical fibers, we see that there is a long exact sequence

$$K_{n+1}(f[t]) \rightarrow K_{n+1}(f[t, 1/t]) \rightarrow K_n\mathbf{Nil}(f) \rightarrow K_n(f[t]) \rightarrow K_n(f[t, 1/t]) \rightarrow$$

and (using Example 5.1) an isomorphism  $\mathbf{Nil}_n(f) \cong NK_{n+1}(f)$ .  $\square$

**Lemma 5.3.** *For any ring homomorphism  $f : A \rightarrow B$ ,  $\mathbf{Nil}_*(f)$  is a graded  $\mathbf{End}_*(A)$ -module.*

*Proof.* A typical object in the Waldhausen category  $S_n\mathbf{Nil}(f)$  is a pair

$$(\mu_*, \nu_*) = ((M_1, \mu_1) \rightarrow \cdots (M_n, \mu_n), (N_0, \nu_0) \rightarrow \cdots (N_n, \nu_n)).$$

There is a pairing  $\mathbf{End}(A) \times S_n\mathbf{Nil}(f) \rightarrow S_n\mathbf{Nil}(f)$  of simplicial Waldhausen categories, sending  $(P, \alpha) \times (\mu_*, \nu_*)$  to

$$((P \otimes M_1, \alpha \otimes \mu_1) \rightarrow \cdots \rightarrow (P \otimes M_n, \alpha \otimes \mu_n), (P \otimes N_0, \alpha \otimes \nu_0) \rightarrow \cdots \rightarrow (P \otimes N_n, \alpha \otimes \nu_n)).$$

It induces a pairing  $K_*\mathbf{End}(A) \otimes K_*\mathbf{Nil}(f) \rightarrow K_*\mathbf{Nil}(f)$ . Since the tensor product  $(\alpha \otimes \beta) \otimes \mu \cong \alpha \otimes (\beta \otimes \mu)$  is associative up to natural isomorphism, the two pairings

$$\mathbf{End}(A) \times \mathbf{End}(A) \times S_n\mathbf{Nil}(f) \rightarrow S_n\mathbf{Nil}(f)$$

agree up to natural isomorphism, making  $K_*\mathbf{Nil}(f)$  a graded  $K_*\mathbf{End}(A)$ -module. In particular,  $\mathbf{Nil}_*(f)$  is a graded module over  $\mathbf{End}_*(A)$ .  $\square$

Recall that the ring  $W(A)$  of big Witt vectors has underlying abelian group  $(1+tA[[t]])^\times$ . Almkvist's theorem [29, II.7.4.3] states that  $[P, \alpha] \mapsto \det(1-t\alpha)$  maps  $\mathbf{End}_0(A)$  isomorphically onto the subring of  $W(A)$  whose underlying abelian group consists of all quotients  $f(t)/g(t)$  of polynomials in  $1+tA[t]$ . The intersection of the ring  $\mathbf{End}_0(A)$  with the ideal  $(1+t^m A[[t]])$  of  $W(A)$  is the ideal  $I_m = \{1+t^m(f/g)\}$

of  $\text{End}_0(A)$ , and  $\text{End}_0(A)/I_m \cong W(A)/(1 + t^m A[[t]])$ . In particular,  $W(A)$  is the completion of  $\text{End}_0(A)$  with respect to the  $t$ -adic filtration.

We say that an  $\text{End}_0(A)$ -module  $M$  is *continuous* if for every  $x \in M$  there is an  $m$  so that  $I_m \cdot x = 0$ . Thus every continuous  $\text{End}_0(A)$ -module  $M$  is also continuous as a  $W(A)$ -module: for every  $x \in M$  we have  $(1 + t^m A[[t]]) \cdot x = 0$  for some  $m$ .

The exact functors  $F_n, V_n : \mathbf{Nil}(A) \rightarrow \mathbf{Nil}(A)$ , defined by  $F_n(P, \nu) = (P, \nu^n)$  and  $V_n(Q, \nu) = (Q[t]/(t^n - \nu), t)$ , commute with  $\mathbf{Nil}(A) \rightarrow \mathbf{Nil}(B)$ . Hence they induce exact endofunctors  $F_n, V_n$  on  $S\mathbf{Nil}(f)$  by  $F_n(\mu_*, \nu_*) = (F_n(\mu_*), F_n(\nu_*))$  and  $V_n(\mu_*, \nu_*) = (V_n(\mu_*), V_n(\nu_*))$ . For  $a \in A$  and  $n > 0$ , and  $\nu$  in  $\text{Nil}_*(f)$ , Almkvist's theorem associates  $(1 - at^n)$  to  $V_n([A, a] - [A, 0])$  and yields the product formula

$$\boxed{\text{W-action}} \quad (5.4) \quad (1 - at^n) * \nu = V_n([A, a] - [A, 0]) * \nu.$$

Stienstra proved in [18, 19] that the  $\text{Nil}_n(A)$  are continuous  $\text{End}_0(A)$ -modules, and hence  $W(A)$ -modules. The key step [18, 2.12] was showing that the projection formula holds:

$$(V_n \alpha) * \nu = V_n(\alpha * F_n(\nu)) \quad \text{for } \alpha \in \text{End}_0(A) \text{ and } \nu \in \text{Nil}_*(A).$$

Here is the corresponding projection formula in the relative setting; we will postpone its proof in order to get to the main result.

$\boxed{\text{projection}}$  **Lemma 5.5.** *For all  $\alpha \in \text{End}_0(A)$  and  $\beta \in \text{Nil}_*(f)$ ,*

$$(V_n \alpha) * \beta = V_n(\alpha * F_n(\beta)).$$

$\boxed{\text{NK}}$  **Theorem 5.6.** *Let  $f : A \rightarrow B$  be a ring map. Then the product (5.4) makes  $\text{Nil}_n(f) \cong NK_{n+1}(f)$  into a continuous  $W(A)$ -module for every integer  $n$ .*

*Proof.* For each  $m > 0$ , let  $\mathbf{Nil}^m(A)$  denote the exact subcategory of all  $(P, \nu)$  in  $\mathbf{Nil}(A)$  such that  $\nu^m = 0$ . Thus we have relative groups  $K_* \mathbf{Nil}^m(f)$  associated to  $K_* \mathbf{Nil}^m(A) \rightarrow K_* \mathbf{Nil}^m(B)$ , and  $K_* \mathbf{Nil}(f)$  is the direct limit of the  $K_* \mathbf{Nil}^m(f)$ .

Suppose that  $n \geq m$ . Clearly,  $F_n$  acts as zero on  $\mathbf{Nil}^m(f)$ . By the projection formula 5.5,  $V_n(\alpha)$  acts as zero on the image  $\text{Nil}_*^m(f)$  of  $K_* \mathbf{Nil}^m(f) \rightarrow K_* \mathbf{Nil}(f) \rightarrow \text{Nil}_*(f)$ . By (5.4),  $(1 - at^n)$  acts as zero on  $\text{Nil}_*^m(f)$ . Since  $\text{Nil}_*(f)$  is the union of the  $\text{Nil}_*^m(f)$ , for any  $\beta \in \text{Nil}_*(f)$  there is an  $m$  such that  $(1 - at^n) \cdot \beta = 0$  for all  $n \geq m$  and  $a \in A$ . This shows that  $\text{Nil}_*(f)$  is a continuous  $\text{End}_0(A)$ -module, and hence a continuous  $W(A)$ -module.  $\square$

*Proof of Lemma 5.5.* Following Stienstra [18, §6], set  $R = \mathbb{Z}[y_1, y_2]$ , and set  $\mathbf{E} = \mathbf{End}(R; S_6)$ , where  $S_6$  is the multiplicative subset of  $R[x]$  generated by  $x$  and  $x^n - y_1^n y_2$ . As pointed out in *loc. cit.*, there is a multi-exact pairing

$$\Theta : \mathbf{E} \times \mathbf{End}(A) \times \mathbf{Nil}(B) \rightarrow \mathbf{Nil}(B)$$

sending  $(E, \omega)$ ,  $(P, \alpha)$  and  $(N, \nu)$  to  $(E \otimes_R (P \otimes_A N), \omega \otimes 1)$ , where  $P \otimes_A N$  is regarded as an  $R$ -module by letting  $y_1$  and  $y_2$  act as  $\alpha \otimes 1$  and  $1 \otimes \nu$ . As this pairing is natural in  $B$ , we may replace  $\mathbf{Nil}(B)$  by  $S.\mathbf{Nil}(f)$ . This yields (among other things) a product

$$\Theta_* : K_0 \mathbf{E} \otimes \text{End}_0(A) \otimes \text{Nil}_*(f) \rightarrow \text{Nil}_*(f).$$

Stienstra proves in *loc. cit.* that the elements  $[R^n, \omega]$  and  $[R^n, \omega']$  agree in  $K_0 \mathbf{E}$ , where

$$\omega = \begin{pmatrix} 0 & & y_1^n y_2 & \\ 1 & & 0 & \\ & \ddots & \vdots & \\ 0 & & 1 & 0 \end{pmatrix} \quad \text{and} \quad \omega' = \begin{pmatrix} 0 & & y_1 y_2 & \\ y_1 & & 0 & \\ & \ddots & \vdots & \\ 0 & & y_1 & 0 \end{pmatrix}.$$

Therefore the two maps

$$\Theta_*([R^n, \omega], -), \Theta_*([R^n, \omega'], -) : \text{End}_0(A) \otimes \text{Nil}_*(f) \rightarrow \text{Nil}_*(f)$$

agree. Stienstra also observes that these maps send  $[P, \alpha] \otimes \beta$  to  $V_n(\alpha * F_n \beta)$  and  $(V_n \alpha) * \beta$ , respectively; see also [19, p.14]. The projection formula follows.  $\square$

## 6. NEGATIVE RELATIVE K-THEORY

Let  $f : X \rightarrow S$  be a morphism of schemes. Then we have a long exact sequence of negative K-groups, part of which is:

$$\boxed{\text{kgroup}} \quad (6.1) \quad \cdots \rightarrow K_{-d}(f) \rightarrow K_{-d}(S) \rightarrow K_{-d}(X) \rightarrow K_{-d-1}(f) \rightarrow K_{-d-1}(S) \rightarrow \cdots$$

$\boxed{\text{vanishing}}$

**Theorem 6.2.** *Let  $f : X \rightarrow S$  be a morphism of  $d$ -dimensional schemes, essentially of finite type over a field  $k$  of characteristic 0. Then for all  $r > 0$ :*

- (1)  $K_n(f) = K_n(f \times \mathbb{A}^r) = 0$  for  $n \leq -d - 2$ .
- (2)  $K_{-d-1}(f) \cong K_{-d-1}(f \times \mathbb{A}^r)$  (“ $f$  is  $K_{-d-1}$ -regular.”)
- (3) If  $f$  is a finite map then  $K_{-d-1}(f) \cong H_{\text{cdh}}^d(S, f_* \mathbb{Z}/\mathbb{Z})$ .

*Proof.* By Corollary 5.9 and Theorem 6.2 of [4],  $K_n(S) \cong K_n(S \times \mathbb{A}^r)$  for all  $n \leq -d$ ,  $K_n(S) = 0$  for  $n < -d$  and  $K_{-d}(S) \cong H_{\text{cdh}}^d(S, \mathbb{Z})$ ; the analogous assertions hold for  $X$ . The exact sequence (6.1) for  $S$  and  $S \times \mathbb{A}^r$  implies the first two assertions. For (3), we have a distinguished triangle cdh sheaves on  $S$ ,

$$\mathbb{Z} \rightarrow f_*\mathbb{Z} \rightarrow f_*\mathbb{Z}/\mathbb{Z} \rightarrow \mathbb{Z}[1].$$

Since the cdh-cohomological dimension of  $S$  is at most  $d$ ,  $H_{\text{cdh}}^{d+1}(S, \mathbb{Z}) = 0$ . Thus the long exact sequence on cdh-cohomology ends in

$$\rightarrow H_{\text{cdh}}^d(S, \mathbb{Z}) \rightarrow H_{\text{cdh}}^d(S, f_*\mathbb{Z}) \rightarrow H_{\text{cdh}}^d(S, f_*\mathbb{Z}/\mathbb{Z}) \rightarrow 0.$$

Since  $f$  is finite, we have  $H_{\text{cdh}}^*(S, f_*\mathbb{Z}) \xrightarrow{\cong} H_{\text{cdh}}^*(X, \mathbb{Z})$ ; assertion (3) follows.  $\square$

*Remark 6.2.1.* Let  $k$  be a perfect field of characteristic  $p$ . Kerz and Strunk have shown in [10] that  $K_n(S)$  is a  $p$ -primary torsion group for  $n < -d$ . Then Theorem 6.2 holds for  $k$  up to  $p$ -torsion.

If in addition  $k$  is a perfect field, over which weak resolution of singularities holds, then Theorem 6.2(1,2) holds for  $k$ . This also follows from [10]; if strong resolution of singularities holds, (1) also follows from the Geisser–Hesselholt theorem in [6] that  $K_n(S) = 0$  for  $n < -d$ .

When  $S$  is a curve, not necessarily defined over  $\mathbb{Q}$ , we have a similar result.

**1dim** **Theorem 6.3.** *Let  $f : X \rightarrow S$  be a finite map of 1-dimensional noetherian schemes. Then  $K_{-1}(f)$  fits into an exact sequence*

$$0 \rightarrow H_{\text{nis}}^1(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) \rightarrow K_{-1}(f) \rightarrow H_{\text{nis}}^0(S, f_*\mathbb{Z}/\mathbb{Z}) \rightarrow 0.$$

*In addition,  $K_{-2}(f) \cong H_{\text{nis}}^1(S, f_*\mathbb{Z}/\mathbb{Z})$  and  $K_n(f) = 0$  for  $n < -2$ .*

*Proof.* By Thomason–Trobaugh [20, 10.8], we have a spectral sequence

$$E_2^{p,q} = H_{\text{nis}}^p(S, \mathcal{K}_{-q}(f)) \Rightarrow K_{-p-q}(f),$$

where  $\mathcal{K}_n(f)$  is the Nisnevich sheafification of the presheaf  $U \mapsto K_n(U, f^{-1}U)$ . Each stalk  $\mathcal{K}_n(f)$  is  $K_n(A, B)$ , where  $A$  is a hensel local ring of dimension  $\leq 1$ . By Lemma 6.4, we have

$$\mathcal{K}_n(f) = \begin{cases} 0 & \text{if } n \leq -2 \\ f_*\mathbb{Z}/\mathbb{Z} & \text{if } n = -1 \\ f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times & \text{if } n = 0. \end{cases}$$

Since  $cd_{\text{nis}}(S) \leq 1$ ,  $E_2^{p,q} \neq 0$  only for  $p = 0, 1$  and  $q \leq 1$ . Thus the spectral sequence degenerates to yield  $K_{-2}(f) \cong H_{\text{nis}}^1(S, f_*\mathbb{Z}/\mathbb{Z})$  and  $K_n(f) = 0$  for  $n < -2$ .  $\square$

**hensel-1dim**

**Lemma 6.4.** *Let  $A$  be a 1-dimensional hensel local ring and  $f : A \hookrightarrow B$  a finite extension. If  $B$  has  $r$  components, then*

$$K_0(f) \cong B^\times/A^\times, \quad K_{-1}(f) \cong \mathbb{Z}^{r-1} \quad \text{and} \quad K_n(f) = 0 \text{ for } n < -1.$$

*Proof.* Since  $B$  is a finite  $A$ -algebra,  $B$  is a finite product of  $r$  hensel local rings. By [24, 2.8],  $K_n(A) = K_n(B) = 0$  for  $n < -1$ . By a result of Drinfeld [29, III.4.4.3], we have  $K_{-1}(A) = K_{-1}(B) = 0$ . The result now follows from (6.1).  $\square$

*Remark 6.5.* A necessary condition for  $K_{-1}(f) = 0$  is that the ring extension  $f : A \hookrightarrow B$  is *anodal*, i.e., if every  $b \in B$  such that  $(b^2 - b) \in A$  and  $(b^3 - b^2) \in A$  belongs to  $A$ . (See [27, 3.1].) This is because (2.3) induces a surjection  $L\det : K_{-1}(f) \rightarrow LPic(f)$ , and we showed in [16] that  $LPic(f) = 0$  implies that  $A \subset B$  is anodal. The converse does not hold, even if  $f$  is a birational extension of domains, as Example 3.5 in [27] shows.

**Example 6.6.** Here is an example to show why we assume  $S$  affine in Proposition 2.5. For each  $n$ , the scheme  $S = \mathbb{P}_k^1$  has a sheaf of algebras  $\mathcal{O}_B = \mathcal{O}_S \oplus \mathcal{O}(n)$  with  $\mathcal{O}(n)$  a square-zero ideal; fix  $n \leq -2$  and set  $X = \text{Spec}(\mathcal{O}_B)$ . Then  $H = H^1(\mathbb{P}^1, \mathcal{O}(n))$  is nonzero and  $\text{Pic}(X) = \text{Pic}(S) \oplus H$ ,  $K_0(X) \cong K_0(S) \oplus H$ . In particular,  $K_{-1}(f) = H \neq 0$ .

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, 1 HOMI BHABA ROAD, COLABA, MUMBAI 400005, INDIA

*E-mail address:* `sadhu@math.tifr.res.in`, `viveksadhu@gmail.com`

MATH. DEPT., RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08901, USA

*E-mail address:* `weibel@math.rutgers.edu`